Extension of the Conley–Zehnder Index and Calculation of the Maslov-Type Index Intervening in Gutzwiller's Trace Formula

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Abstract

The aim of this paper is to define and study a non-trivial extension of the Conley–Zehnder index. We make use for this purpose of the global Arnol'd–Leray–Maslov index constructed by the authors in previous work. This extension allows us to obtain explicit formulae for the calculation of the Gutzwiller–Maslov index of a Hamiltonian periodic orbit. We show that this index is related to the usual Maslov index via Morse's index of concavity of a periodic Hamiltonian orbit. In addition we prove a formula allowing to calculate the index of a repeated orbit.

1 Introduction

This paper is devoted to a redefinition and extension of the Conley–Zehnder index [3, 13] from the theory of periodic Hamiltonian orbits. This will allow us to prove two new (and useful) formulae. The first of these formulae expresses the index of the product of two symplectic paths in terms of the *symplectic Cayley transform* of a symplectic matrix; the second shows that the difference between the Conley–Zehnder index and the usual Maslov index is (up to the sign) *Morse's index of concavity.*

Besides the intrinsic interest of our constructions (they allow practical calculations of the Conley–Zehnder index even in degenerate cases) they probably settle once for all some well-known problems from Gutzwiller's [12] quantization scheme of classically chaotic systems. Let us briefly recall the idea. Let $\widehat{H}$ be a Hamiltonian operator with discrete spectrum $E_1, E_2, ...$ One wants to find an asymptotic expression, for $\hbar \to 0$, of the "level density"

$$d(E) = \sum_{k=1}^{\infty} \delta(E - E_j) = -\frac{1}{\pi} \lim_{\varepsilon \to 0+} \operatorname{Tr} G(x, x', E + i\varepsilon)$$

(G the Green function determined by $\widehat{H}$). Writing $d(E) = \bar{d}(E) + \widetilde{d}(E)$ where $\bar{d}(E)$ is the Thomas–Fermi or smoothed density of states, and $\widetilde{d}(E)$ the "oscillating term", Gutzwiller's formula says that in the limit $\hbar \to 0$ we have

$$\widetilde{d}(E) = \widetilde{d}_{\text{Gutz}}(E) + O(\hbar)$$

where

$$\widetilde{d}_{\text{Gutz}}(E) = \frac{1}{\pi\hbar} \operatorname{Re} \sum_{\gamma} \frac{T_\gamma i^{\mu_\gamma}}{\sqrt{|\det(\tilde{S}_\gamma - I)|}} \exp\left(\frac{i}{\hbar} \oint_\gamma p \, dx\right). \tag{1}$$

The sum in the right-hand side of the formula above is taken over all *periodic orbits* γ with period T_γ of the classical Hamiltonian H, *including their repetitions*; the exponent μ_γ is an integer and and $\tilde{S}_\gamma$ is the stability matrix of γ. Gutzwiller's theory is far from being fully understood (there are problems due to possible divergences of the series, insufficient error estimates, etc.). We will not discuss these delicate problems here; what we will do is instead to focus on the integers μ_γ, to which much literature has been devoted (see for instance [1, 4, 21] and the references therein). As is well-known, μ_γ is not the usual Maslov index familiar from EBK quantization of Lagrangian manifolds, but rather (up to the sign) the Conley–Zehnder index of the orbit γ:

"Gutzwiller–Maslov index" $= -$(Conley–Zehnder index)

2

(see *e.g.* Muratore-Ginnaneschi in [18]; in [11] one of us has given a different proof using an ingenious and promising derivation of Gutzwiller's formula by Mehlig and Wilkinson [16] based on the Weyl representation of metaplectic operators). We notice that Sugita [23] has proposed a scheme for the calculation of what is essentially the Conley–Zehnder index using a Feynman path-integral formalism; his constructions are however mathematically illegitimate and seem difficult to justify without using deep results from functional analysis (but the effort might perhaps be worthwhile).

This paper is structured as follows:

1. Redefine the Conley–Zehnder index in terms of globally defined indices (the Wall–Kashiwara index and the Arnol'd–Leray–Maslov index); we will thus obtain a non-trivial extension ν of that index which is explicitly computable for all paths, even in the case $\det(S - I) = 0$; this is useful in problems where degeneracies arise (for instance the isotropic harmonic oscillator, see [20] and the example of last section in the present paper);

2. We will prove a simple formula for the Conley–Zehnder index of the product of two symplectic paths. We will see that in particular the index of an orbit which is repeated r times is

$$\nu_{\gamma_r} = r\nu_\gamma + \tfrac{1}{2}(r - 1)\,\mathrm{sign}\,M_S$$

   where

$$M_S = \frac{1}{2}J(S + I)(S - I)^{-1}$$

   is the symplectic Cayley transform of the monodromy matrix S;

3. We will finally prove that the Conley–Zehnder index ν_γ of a non-degenerate periodic orbit γ is related to its Maslov index m_γ by the simple formula

$$\nu_\gamma = m_\gamma - \mathrm{Inert}\,W_\gamma$$

   where $\mathrm{Inert}\,W_\gamma$ is Morse's "index of concavity" [17, 18], defined in terms of the generating function of the monodromy matrix.

We close this article by performing explicit calculations in the case of the two-dimensional anisotropic harmonic oscillator; this allows us to recover a formula obtained by non-rigorous methods in the literature.

Notations

We will denote by σ the standard symplectic form on $\mathbb{R}^{2n} = \mathbb{R}^n_x \times \mathbb{R}^n_p$:

$$\sigma(z, z') = \langle p, x' \rangle - \langle p', x \rangle \quad \text{if} \quad z = (x, p), \, z' = (x'p')$$

that is, in matrix form

$$\sigma(z, z') = \langle Jz, z' \rangle \quad , \quad J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The real symplectic group $\mathrm{Sp}(n)$ consists of all linear automorphisms S of $\mathbb{R}^{2n}$ such that $\sigma(Sz, Sz') = \sigma(z, z')$ for all z, z'. Equivalently:

$$S \in \mathrm{Sp}(n) \Longleftrightarrow S^T J S = S J S^T = J.$$

$\mathrm{Sp}(n)$ is a connected Lie group and $\pi_1[\mathrm{Sp}(n)] \equiv (\mathbb{Z}, +)$. We denote by $\mathrm{Lag}(n)$ the Lagrangian Grassmannian of $(\mathbb{R}^{2n}, \sigma)$, that is: $\ell \in \mathrm{Lag}(n)$ if and only ℓ is a n-plane in $\mathbb{R}^{2n}$ on which σ vanishes identically. We will write $\ell_X = \mathbb{R}^n_x \times 0$ and $\ell_P = 0 \times \mathbb{R}^n_p$ (the "horizontal" and "vertical" polarizations).

2 Prerequisites

In this section we review previous results [7, 8, 10] on Lagrangian and symplectic Maslov indices generalizing those of Leray [14]. An excellent comparative study of the indices used here with other indices appearing in the literature can be found in Cappell *et al.* [2].

In what follows (E, ω) is a finite-dimensional symplectic space, $\dim E = 2n$, and $\mathrm{Sp}(E, \omega)$, $\mathrm{Lag}(E, \omega)$ the associated symplectic group and Lagrangian Grassmannian.

2.1 The Wall–Kashiwara index

Let (ℓ, ℓ', ℓ'') be a triple of elements of $\mathrm{Lag}(E, \omega)$; by definition [2, 15, 24] the Wall–Kashiwara index (or: signature) $\tau(\ell, \ell', \ell'')$ is the signature of the quadratic form

$$Q(z, z', z'') = \sigma(z, z') + \sigma(z', z'') + \sigma(z'', z')$$

on $\ell \oplus \ell' \oplus \ell''$. The index τ is antisymmetric:

$$\tau(\ell, \ell', \ell'') = -\tau(\ell', \ell, \ell'') = -\tau(\ell, \ell'', \ell') = -\tau(\ell'', \ell', \ell);$$

4

it is a symplectic invariant:

$$\tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'') \ \text{ for } S \in \mathrm{Sp}(n)$$

and it has the following essential cocycle property:

$$\tau(\ell, \ell', \ell'') - \tau(\ell', \ell'', \ell''') + \tau(\ell', \ell'', \ell''') - \tau(\ell', \ell'', \ell''') = 0. \qquad (2)$$

Moreover its values modulo 2 are given by the formula:

$$\tau(\ell, \ell', \ell'') \equiv n + \dim \ell \cap \ell' + \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell \mod 2. \qquad (3)$$

Let $(E, \omega) = (E' \oplus E'', \omega' \oplus \omega'')$; identifying $\mathrm{Lag}(E', \omega') \oplus \mathrm{Lag}(E'', \omega'')$ with a subset of $\mathrm{Lag}(E, \omega)$ we have the following additivity formula:

$$\tau(\ell_1 \oplus \ell_2, \ell_1' \oplus \ell_2', \ell_1'' \oplus \ell_2'') = \tau_1(\ell_1, \ell_1', \ell_1'') + \tau_2(\ell_2, \ell_2', \ell_2'')$$

where τ_1 and τ_2 are the signatures on $\mathrm{Lag}(E', \omega')$ and $\mathrm{Lag}(E'', \omega'')$.

The following Lemma will be helpful in our study of the Conley–Zehnder index:

Lemma 1 *(i) If $\ell \cap \ell'' = 0$ then $\tau(\ell, \ell', \ell'')$ is the signature of the quadratic form*

$$Q'(z') = \omega(\mathrm{Pr}_{\ell\ell''} z', z') = \omega(z', \mathrm{Pr}_{\ell''\ell} z')$$

on ℓ', where $\mathrm{Pr}_{\ell\ell''}$ is the projection onto ℓ along ℓ'' and $\mathrm{Pr}_{\ell''\ell} = I - \mathrm{Pr}_{\ell\ell''}$ is the projection on ℓ'' along ℓ. (ii) Let (ℓ, ℓ', ℓ'') be a triple of Lagrangian planes such that an $\ell = \ell \cap \ell' + \ell \cap \ell''$. Then $\tau(\ell, \ell', \ell'') = 0$.

(See *e.g.* [15] for a proof).

The index of inertia of the triple (ℓ, ℓ', ℓ'') is defined by

$$\mathrm{Inert}(\ell, \ell', \ell'') = \frac{1}{2}(\tau(\ell, \ell', \ell'') + n + \dim \ell \cap \ell' - \dim \ell' \cap \ell'' + \dim \ell'' \cap \ell); \quad (4)$$

in view of (3) it is an integer. When the Lagrangian planes ℓ, ℓ', ℓ'' are pairwise transverse it follows from the first part of Lemma 1 that $\mathrm{Inert}(\ell, \ell', \ell'')$ coincides with the index of inertia defined by Leray [14]: see [7, 8].

2.2 The ALM index

Recall [7, 8] (also [10] for a review and [9] for calculations in the case $n = 1$) that the ALM (=Arnol'd–Leray–Maslov) index on the universal covering $\mathrm{Lag}_\infty(E, \omega)$ of $\mathrm{Lag}(E, \omega)$ is the unique mapping

$$\mu : (\mathrm{Lag}_\infty(E, \omega))^2 \longrightarrow \mathbb{Z}$$

having the two following properties:

- μ is locally constant on each set $\{(\ell_\infty, \ell'_\infty) : \dim \ell \cap \ell' = k\}$ $(0 \le k \le n)$;

- For all ℓ_∞, ℓ'_∞, ℓ''_∞ in $\mathrm{Lag}_\infty(E, \omega)$ with projections ℓ, ℓ', ℓ'' we have

$$\mu(\ell_\infty, \ell'_\infty) - \mu(\ell_\infty, \ell''_\infty) + \mu(\ell'_\infty, \ell''_\infty) = \tau(\ell, \ell', \ell'') \tag{5}$$

where τ is the Wall–Kashiwara index on $\mathrm{Lag}(E, \omega)$.

The ALM index has in addition the following properties:

$$\mu(\ell_\infty, \ell'_\infty) \equiv n + \dim \ell \cap \ell' \mod 2 \tag{6}$$

$(n = \frac{1}{2} \dim E)$ and

$$\mu(\beta^r \ell_\infty, \beta^{r'} \ell'_\infty) = \mu(\ell_\infty, \ell'_\infty) + 2(r - r') \tag{7}$$

for all integers r and r'; here β denotes the generator of $\pi_1[\mathrm{Lag}(E, \omega)] \equiv (\mathbb{Z}, +)$ whose image in $\mathbb{Z}$ is $+1$. From the dimensional additivity property of the signature τ immediately follows that if $\ell_{1,\infty} \oplus \ell_{2,\infty}$ and $\ell'_{1,\infty} \oplus \ell'_{2,\infty}$ are in

$$\mathrm{Lag}_\infty(E', \omega') \oplus \mathrm{Lag}_\infty(E'', \omega'') \subset \mathrm{Lag}_\infty(E, \omega)$$

then

$$\mu(\ell_{1,\infty} \oplus \ell_{2,\infty}, \ell'_{1,\infty} \oplus \ell'_{2,\infty}) = \mu'(\ell_{1,\infty}, \ell'_{1,\infty}) + \mu''(\ell_{2,\infty}, \ell'_{2,\infty}) \tag{8}$$

where μ' and μ'' are the ALM indices on $\mathrm{Lag}_\infty(E', \omega')$ and $\mathrm{Lag}_\infty(E'', \omega'')$, respectively.

When (E, ω) is the standard symplectic space $(\mathbb{R}^{2n}, \sigma)$ the "Souriau mapping" [22] identifies $\mathrm{Lag}(E, \omega) = \mathrm{Lag}(n)$ with the set

$$\mathrm{W}(n, \mathbb{C}) = \{w \in \mathrm{U}(n, \mathbb{C}) : w = w^T\}$$

of symmetric unitary matrices. This is done by associating to to $\ell = u\ell_P$ $(u \in \mathrm{U}(n, \mathbb{C}))$ the matrix $w = uu^T$; the Maslov bundle $\mathrm{Lag}_\infty(n)$ is then identified with

$$W_\infty(n, \mathbb{C}) = \{(w, \theta) : w \in \mathrm{W}(n, \mathbb{C}), \ \det w = e^{i\theta}\};$$

the projection $\pi^{\mathrm{Lag}} : \ell_\infty \longmapsto \ell$ becoming $(w, \theta) \longmapsto w$. The ALM index is then calculated as follows:

- If $\ell \cap \ell' = 0$ then

$$\mu(\ell_\infty, \ell'_\infty) = \frac{1}{\pi} \left[\theta - \theta' + i \operatorname{Tr} \operatorname{Log}(-w(w')^{-1}) \right] \tag{9}$$

  (the transversality condition $\ell \cap \ell'$ is equivalent to $-w(w')^{-1}$ having no negative eigenvalue);

- If $\ell \cap \ell' \neq 0$ one chooses any ℓ'' such that $\ell \cap \ell'' = \ell' \cap \ell'' = 0$ and one then calculates $\mu(\ell_\infty, \ell'_\infty)$ using the formula (5), the values of $\mu(\ell_\infty, \ell''_\infty)$ and $\mu(\ell'_\infty, \ell''_\infty)$ being given by (9). (The cocycle property (2) of τ guarantees that the result does not depend on the choice of ℓ'', see [7, 8]).

2.3   The relative Maslov indices on $\mathrm{Sp}(E, \omega)$

We begin by recalling the definition of the Maslov index for loops in $\mathrm{Sp}(n)$. Let γ be a continuous mapping $[0, 1] \longrightarrow \mathrm{Sp}(n)$ such that $\gamma(0) = \gamma(1)$, and set $\gamma(t) = S_t$. Then $U_t = (S_t S_t)^{-1/2} S_t$ is the orthogonal part in the polar decomposition of S_t:
$$U_t \in \mathrm{Sp}(n) \cap \mathrm{O}(2n, \mathbb{R}).$$

Let us denote by u_t the image $\iota(U_t)$ of U_t in $\mathrm{U}(n, \mathbb{C})$:

$$\iota(U_t) = A + iB \ \text{ if } \ U = \begin{bmatrix} A & -B \\ B & A \end{bmatrix}$$

and define $\rho(S_t) = \det u_t$. The Maslov index of γ is by definition the degree of the loop $t \longmapsto \rho(S_t)$ in S^1:

$$m(\gamma) = \deg[t \longmapsto \det(\iota(U_t))] \ , \ 0 \leq t \leq 1.$$

Let α be the generator of $\pi_1[\mathrm{Sp}(E,\omega)] \equiv (\mathbb{Z},+)$ whose image in $\mathbb{Z}$ is $+1$; if γ is homotopic to α^r then

$$m(\gamma) = m(\alpha^r) = 2r. \tag{10}$$

The definition of the Maslov index can be extended to arbitrary paths in $\mathrm{Sp}(E,\omega)$ using the properties of the ALM index. This is done as follows: let $\ell = \pi^{\mathrm{Lag}}(\ell_\infty) \in \mathrm{Lag}(E,\omega)$; we define the Maslov index of $S_\infty \in \mathrm{Sp}_\infty(E,\omega)$ relative to ℓ by

$$\mu_\ell(S_\infty) = \mu(S_\infty \ell_\infty, \ell_\infty); \tag{11}$$

one shows (see [7, 8]) that the right-hand side only depends on the projection ℓ of ℓ_∞, justifying the notation.

Here are three fundamental properties of the relative Maslov index; we will need all of them to study the Conley–Zehnder index:

- *Product*: For all S_∞, S'_∞ in $\mathrm{Sp}_\infty(E,\omega)$ we have

$$\mu_\ell(S_\infty S'_\infty) = \mu_\ell(S_\infty) + \mu_\ell(S'_\infty) + \tau(\ell, S\ell, SS'\ell); \tag{12}$$

- *Action of* $\pi_1[\mathrm{Sp}(n)]$: We have

$$\mu_\ell(\alpha^r S_\infty) = \mu_\ell(S_\infty) + 4r \tag{13}$$

  for all $r \in \mathbb{Z}$;

- *Topological property*: The mapping $(S_\infty, \ell) \longmapsto \mu_\ell(S_\infty)$ is locally constant on each of the sets

$$\{(S_\infty, \ell) : \dim S\ell \cap \ell = k\} \subset \mathrm{Sp}_\infty(E,\omega) \times \mathrm{Lag}(E,\omega) \tag{14}$$

$(0 \leq k \leq n)$.

The two first properties readily follow from, respectively, (5) and (7). The third follows from the fact that the ALM index is locally constant on the sets $\{(\ell_\infty, \ell'_\infty) : \dim \ell \cap \ell' = k\}$. Note that (13) implies that

$$\mu_\ell(\alpha^r) = 4r$$

hence the restriction of any of the μ_ℓ to loops γ in $\mathrm{Sp}(E,\omega)$ is *twice* the Maslov index $m(\gamma)$ defined above; it is therefore sometimes advantageous to use the variant of μ_ℓ defined by

$$m_\ell(S_\infty) = \frac{1}{2}(\mu_\ell(S_\infty) + n + \dim(S\ell \cap \ell)) \tag{15}$$

8

where $n = \frac{1}{2} \dim E$. We will call $m_\ell(S_\infty)$ the *reduced* (relative) Maslov index. In view of property (6) it is an integer; the properties of m_ℓ are obtained, *mutatis mutandis*, from those of μ_ℓ; for instance property (12) becomes

$$m_\ell(S_\infty S'_\infty) = m_\ell(S_\infty) + m_\ell(S'_\infty) + \text{Inert}(\ell, S\ell, SS'\ell)$$

where Inert is the index of inertia defined by (4).

It follows from the cocycle property of the signature τ that the Maslov indices corresponding to two choices ℓ and ℓ' are related by the formula

$$\mu_\ell(S_\infty) - \mu_{\ell'}(S_\infty) = \tau(S\ell, \ell, \ell') - \tau(S\ell, S\ell', \ell'); \tag{16}$$

similarly

$$m_\ell(S_\infty) - m_{\ell'}(S_\infty) = \text{Inert}(S\ell, \ell, \ell') - \text{Inert}(S\ell, S\ell', \ell'). \tag{17}$$

Assume that $(E, \omega) = (E' \oplus E'', \omega' \oplus \omega'')$ and $\ell' \in \text{Lag}(E', \omega')$, $\ell'' \in \text{Lag}(E'', \omega'')$; the additivity property (8) of the ALM index implies that if $S'_\infty \in \text{Sp}_\infty(E', \omega')$, $S''_\infty \in \text{Sp}_\infty(E'', \omega'')$ then

$$\mu_{\ell' \oplus \ell''}(S'_\infty \oplus S''_\infty) = \mu_{\ell'}(S'_\infty) + \mu_{\ell_2}(S''_\infty) \tag{18}$$

where $\text{Sp}_\infty(E', \omega') \oplus \text{Sp}_\infty(E'', \omega'')$ is identified in the obvious way with a subgroup of $\text{Sp}_\infty(E, \omega)$; a similar property holds for the reduced index m_ℓ.

3 The index ν on $\text{Sp}_\infty(n)$

In this section we define and study a function $\nu : \text{Sp}_\infty(n) \longrightarrow \mathbb{Z}$ extending the Conley–Zehnder index [3]. We begin by recalling the definition and main properties of the latter.

3.1 Review of the Conley–Zehnder index

Let Σ be a continuous path $[0, 1] \longrightarrow \text{Sp}(n)$ such that $\Sigma(0) = I$ and $\det(\Sigma(1) - I) \neq 0$. Loosely speaking, the Conley–Zehnder index [3] counts algebraically the number of points in the open interval $]0, 1[$ for which $\Sigma(t)$ has 1 as an eigenvalue. To give a more precise definition we need some notations. Let us define three subsets of $\text{Sp}(n)$ by

$$\text{Sp}_0(n) = \{S : \det(S - I) = 0\}$$
$$\text{Sp}^+(n) = \{S : \det(S - I) > 0\}$$
$$\text{Sp}^-(n) = \{S : \det(S - I) < 0\}.$$

These sets partition $\mathrm{Sp}(n)$, and $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$ are moreover arcwise connected; the symplectic matrices $S^+ = -I$ and

$$S^- = \begin{bmatrix} L & 0 \\ 0 & L^{-1} \end{bmatrix} \ , \ L = \mathrm{diag}[2, -1, ..., -1]$$

belong to $\mathrm{Sp}^+(n)$ and $\mathrm{Sp}^-(n)$, respectively.

Let us now denote by $C_\pm(2n, \mathbb{R})$ the space of all paths $\Sigma : [0, 1] \longrightarrow \mathrm{Sp}(n)$ with $\Sigma(0) = I$ and $\Sigma(1) \in \mathrm{Sp}^\pm(n)$. Any such path can be extended into a path $\widetilde{\Sigma} : [0, 2] \longrightarrow \mathrm{Sp}(n)$ such that $\widetilde{\Sigma}(t) \in \mathrm{Sp}^\pm(n)$ for $1 \leq t \leq 2$ and $\widetilde{\Sigma}(2) = S^+$ or $\widetilde{\Sigma}(2) = S^-$. Let ρ be the mapping $\mathrm{Sp}(n) \longrightarrow S^1$, $\rho(S_t) = \det u_t$, used in the definition of the Maslov index for symplectic loops. The Conley–Zehnder index of Σ is, by definition, the winding number of the loop $(\rho \circ \widetilde{\Sigma})^2$ in S^1:

$$i_{\mathrm{CZ}}(\Sigma) = \deg[t \longmapsto (\rho(\widetilde{\Sigma}(t)))^2, \ 0 \leq t \leq 2].$$

It turns out that $i_{\mathrm{CZ}}(\Sigma)$ is invariant under homotopy as long as the end-point $S = \Sigma(1)$ remains in $\mathrm{Sp}^\pm(n)$; in particular it does not change under homotopies with fixed endpoints so we may view i_{CZ} as defined on the subset

$$\mathrm{Sp}_\infty^*(n) = \{S_\infty : \det(S - I) \neq 0\}$$

of the universal covering group $\mathrm{Sp}_\infty(n)$. With this convention one proves [13] that the Conley-Zehnder index is the unique mapping $i_{\mathrm{CZ}} : \mathrm{Sp}_\infty^*(n) \longrightarrow \mathbb{Z}$ having the following properties:

(CZ₁) *Antisymmetry*: For every S_∞ we have

$$i_{\mathrm{CZ}}(S_\infty^{-1}) = -i_{\mathrm{CZ}}(S_\infty)$$

where S_∞^{-1} is the homotopy class of the path $t \longmapsto S_t^{-1}$;

(CZ₂) *Continuity*: Let Σ be a symplectic path representing S_∞ and Σ' a path joining S to an element S' belonging to the same component $\mathrm{Sp}^\pm(n)$ as S. Let S'_∞ be the homotopy class of $\Sigma * \Sigma'$. We have

$$i_{\mathrm{CZ}}(S_\infty) = i_{\mathrm{CZ}}(S'_\infty);$$

(CZ₃) *Action of* $\pi_1[\mathrm{Sp}(n)]$:

$$i_{\mathrm{CZ}}(\alpha^r S_\infty) = i_{\mathrm{CZ}}(S_\infty) + 2r$$

for every $r \in \mathbb{Z}$.

We observe that these three properties are characteristic of the Conley–Zehnder index in the sense that any other function $i'_{\text{CZ}} : \text{Sp}^*_\infty(n) \longrightarrow \mathbb{Z}$ satisfying then must be identical to i_{CZ}. Set in fact $\delta = i_{\text{CZ}} - i'_{\text{CZ}}$. In view of (CZ$_3$) we have $\delta(\alpha^r S_\infty) = \delta(S_\infty)$ for all $r \in \mathbb{Z}$ hence δ is defined on $\text{Sp}^*(n) = \text{Sp}^+(n) \cup \text{Sp}^-(n)$ so that $\delta(S_\infty) = \delta(S)$ where $S = S_1$, the endpoint of the path $t \longmapsto S_t$. Property (CZ$_2$) implies that this function $\text{Sp}^*(n) \longrightarrow \mathbb{Z}$ is constant on both $\text{Sp}^+(n)$ and $\text{Sp}^-(n)$. We next observe that since $\det S = 1$ we have $\det(S^{-1} - I) = \det(S - I)$ so that S and S^{-1} always belong to the same set $\text{Sp}^+(n)$ or $\text{Sp}^-(n)$ if $\det(S - I) \neq 0$. Property (CZ$_1$) then implies that δ must be zero on both $\text{Sp}^+(n)$ or $\text{Sp}^-(n)$.

Two other noteworthy properties of the Conley–Zehnder are:

(CZ$_4$) *Normalization*: Let J_1 be the standard symplectic matrix in $Sp(1)$. If S_1 is the path $t \longrightarrow e^{\pi t J_1}$ ($0 \leq t \leq 1$) joining I to $-I$ in $\text{Sp}(1)$ then $i_{\text{CZ},1}(S_{1,\infty}) = 1$ ($i_{\text{CZ},1}$ the Conley–Zehnder index on $\text{Sp}(1)$);

(CZ$_5$) *Dimensional additivity*: if $S_{1,\infty} \in \text{Sp}^*_\infty(n_1)$, $S_{2,\infty} \in \text{Sp}^*_\infty(n_2)$, $n_1 + n_2 = n$, then

$$i_{\text{CZ}}(S_{1,\infty} \oplus S_{2,\infty}) = i_{\text{CZ},1}(S_{1,\infty}) + i_{\text{CZ},2}(S_{2,\infty})$$

where $i_{\text{CZ},j}$ is the Conley–Zehnder index on $\text{Sp}(n_j)$, $j = 1, 2$.

3.2 Symplectic Cayley transform

Our extension of the index i_{CZ} requires a notion of Cayley transform for symplectic matrices. If $S \in \text{Sp}(n)$, $\det(S - I) \neq 0$, we call the matrix

$$M_S = \frac{1}{2}J(S + I)(S - I)^{-1} \tag{19}$$

the "symplectic Cayley transform of S". Equivalently:

$$M_S = \frac{1}{2}J + J(S - I)^{-1}. \tag{20}$$

It is straightforward to check that M_S always is a symmetric matrix: $M_S = M_S^T$ (it suffices for this to use the fact that $S^T J S = S J S^T = J$).

The symplectic Cayley transform has in addition the following properties, which are interesting by themselves:

Lemma 2 *(i) We have*

$$(M_S + M_{S'})^{-1} = -(S' - I)(SS' - I)^{-1}(S - I)J \tag{21}$$

11

and the symplectic Cayley transform of the product SS' is (when defined) given by the formula

$$M_{SS'} = M_S + (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S - I)^{-1}. \qquad (22)$$

(ii) The symplectic Cayley transform of S and S^{-1} are related by

$$M_{S^{-1}} = -M_S. \qquad (23)$$

Proof. (i) We begin by noting that (20) implies that

$$M_S + M_{S'} = J(I + (S - I)^{-1} + (S' - I)^{-1}) \qquad (24)$$

hence the identity (21). In fact, writing $SS' - I = S(S' - I) + S - I$, we have

$$
\begin{aligned}
(S' - I)(SS' - I)^{-1}(S - I) &= (S' - I)(S(S' - I) + S - I)^{-1}(S - I) \\
&= ((S - I)^{-1}S(S' - I)(S' - I)^{-1} + (S' - I)^{-1})^{-1} \\
&= ((S - I)^{-1}S + (S' - I)^{-1}) \\
&= I + (S - I)^{-1} + (S' - I)^{-1};
\end{aligned}
$$

the equality (21) follows in view of (24). Let us prove (22); equivalently

$$M_S + M = M_{SS'} \qquad (25)$$

where M is the matrix defined by

$$M = (S^T - I)^{-1}J(M_S + M_{S'})^{-1}J(S - I)^{-1}$$

that is, in view of (21),

$$M = (S^T - I)^{-1}J(S' - I)(SS' - I)^{-1}.$$

Using the obvious relations $S^T = -JS^{-1}J$ and $(-S^{-1} + I)^{-1} = S(S - I)^{-1}$ we have

$$
\begin{aligned}
M &= (S^T - I)^{-1}J(S' - I)(SS' - I)^{-1} \\
&= -J(-S^{-1} + I)^{-1}(S' - I)(SS' - I)^{-1} \\
&= -JS(S - I)^{-1}(S' - I)(SS' - I)^{-1}
\end{aligned}
$$

that is, writing $S = S - I + I$,

$$M = -J(S' - I)(SS' - I)^{-1} - J(S - I)^{-1}(S' - I)(SS' - I)^{-1}.$$

Replacing M_S by its value (20) we have

$$M_S + M =$$
$$J(\tfrac{1}{2}I + (S - I)^{-1} - (S' - I)(SS' - I)^{-1} - (S - I)^{-1}(S' - I)(SS' - I)^{-1});$$

noting that

$$(S - I)^{-1} - (S - I)^{-1}(S' - I)(SS' - I)^{-1} =$$
$$(S - I)^{-1}(SS' - I - S' + I)(SS' - I)^{-1})$$

that is

$$(S - I)^{-1} - (S - I)^{-1}(S' - I)(SS' - I)^{-1} = (S - I)^{-1}(SS' - S')(SS' - I)^{-1}$$
$$= S'(SS' - I)^{-1})$$

we get

$$M_S + M = J(\tfrac{1}{2}I - (S' - I)(SS' - I)^{-1} + S'(SS' - I)^{-1})$$
$$= J(\tfrac{1}{2}I + (SS' - I)^{-1})$$
$$= M_{SS'}$$

which we set out to prove. (ii) Formula (23) follows from the sequence of equalities

$$M_{S^{-1}} = \tfrac{1}{2}J + J(S^{-1} - I)^{-1}$$
$$= \tfrac{1}{2}J - JS(S - I)^{-1}$$
$$= \tfrac{1}{2}J - J(S - I + I)(S - I)^{-1}$$
$$= -\tfrac{1}{2}J - J(S - I)^{-1}$$
$$= -M_S.$$

∎

3.3 Definition and properties of $\nu(S_\infty)$

We define on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ a symplectic form $\sigma^\ominus$ by

$$\sigma^\ominus(z_1, z_2; z_1', z_2') = \sigma(z_1, z_1') - \sigma(z_2, z_2')$$

and denote by $\mathrm{Sp}^\ominus(2n)$ and $\mathrm{Lag}^\ominus(2n)$ the corresponding symplectic group and Lagrangian Grassmannian. Let $\mu^\ominus$ be the ALM index on $\mathrm{Lag}_\infty^\ominus(2n)$ and $\mu_L^\ominus$ the Maslov index on $\mathrm{Sp}_\infty^\ominus(2n)$ relative to $L \in \mathrm{Lag}^\ominus(2n)$.

For $S_\infty \in \mathrm{Sp}_\infty(n)$ we define

$$\nu(S_\infty) = \frac{1}{2}\mu^\ominus((I \oplus S)_\infty \Delta_\infty, \Delta_\infty) \qquad (26)$$

where $(I \oplus S)_\infty$ is the homotopy class in $\mathrm{Sp}^\ominus(2n)$ of the path

$$t \longmapsto \{(z, S_t z) : z \in \mathbb{R}^{2n}\} \ , \quad 0 \le t \le 1$$

and $\Delta = \{(z, z) : z \in \mathbb{R}^{2n}\}$ the diagonal of $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. Setting $S_t^\ominus = I \oplus S_t$ we have $S_t^\ominus \in \mathrm{Sp}^\ominus(2n)$ hence formulae (26) is equivalent to

$$\nu(S_\infty) = \frac{1}{2}\mu_\Delta^\ominus(S_\infty^\ominus) \qquad (27)$$

where $\mu_\Delta^\ominus$ is the relative Maslov index on $\mathrm{Sp}_\infty^\ominus(2n)$ corresponding to the choice $\Delta \in \mathrm{Lag}^\ominus(2n)$.

Note that replacing n by $2n$ in the congruence (6) we have

$$\mu^\ominus((I \oplus S)_\infty \Delta_\infty, \Delta_\infty) \equiv \dim((I \oplus S)\Delta \cap \Delta) \mod 2$$
$$\equiv \dim \mathrm{Ker}(S - I) \mod 2$$

and hence

$$\nu(S_\infty) \equiv \frac{1}{2}\dim \mathrm{Ker}(S - I) \mod 1.$$

Since the eigenvalue 1 of S has even multiplicity $\nu(S_\infty)$ is thus always an integer.

The index ν has the following three important properties; the third is essential for the calculation of the index of repeated periodic orbits (it clearly shows that ν is not in general additive):

Proposition 3 *(i) For all $S_\infty \in \mathrm{Sp}_\infty(n)$ we have*

$$\nu(S_\infty^{-1}) = -\nu(S_\infty) \ , \quad \nu(I_\infty) = 0 \qquad (28)$$

(I_∞ the identity of the group $\mathrm{Sp}_\infty(n)$). (ii) For all $r \in \mathbb{Z}$ we have

$$\nu(\alpha^r S_\infty) = \nu(S_\infty) + 2r \ , \quad \nu(\alpha^r) = 2r \qquad (29)$$

(iii) Let S_∞ be the homotopy class of a path Σ in $\mathrm{Sp}(n)$ joining the identity to $S \in \mathrm{Sp}^*(n)$, and let $S' \in \mathrm{Sp}(n)$ be in the same connected component $\mathrm{Sp}^\pm(n)$ as S. Then $\nu(S'_\infty) = \nu(S_\infty)$ where S'_∞ is the homotopy class in $\mathrm{Sp}(n)$ of the concatenation of Σ and a path joining S to S' in $\mathrm{Sp}_0(n)$.

Proof. (i) Formulae (28) immediately follows from the equality $(S_\infty^\ominus)^{-1} = (I \oplus S^{-1})_\infty$ and the antisymmetry of $\mu_\Delta^\ominus$. (ii) The second formula (29) follows from the first using (28). To prove the first formula (29) it suffices to observe that to the generator α of $\pi_1[\mathrm{Sp}(n)]$ corresponds the generator $I_\infty \oplus \alpha$ of $\pi_1[\mathrm{Sp}^\ominus(2n)]$; in view of property (13) of the Maslov index it follows that

$$\nu(\alpha^r S_\infty) = \frac{1}{2}\mu_\Delta^\ominus((I_\infty \oplus \alpha)^r S_\infty^\ominus)$$

$$= \frac{1}{2}(\mu_\Delta^\ominus(S_\infty^\ominus) + 4r)$$

$$= \nu(S_\infty) + 2r.$$

(iii) Assume in fact that S and S' belong to, say, $\mathrm{Sp}^+(n)$. Let S_∞ be the homotopy class of the path Σ, and Σ' a path joining S to S' in $\mathrm{Sp}^+(n)$ (we parametrize both paths by $t \in [0,1]$). Let $\Sigma'_{t'}$ be the restriction of Σ' to the interval $[0, t']$, $t' \leq t$ and $S_\infty(t')$ the homotopy class of the concatenation $\Sigma * \Sigma'_{t'}$. We have $\det(S(t) - I) > 0$ for all $t \in [0, t']$ hence $S_\infty^\ominus(t)\Delta \cap \Delta \neq 0$ as t varies from 0 to 1. It follows from the fact that the $\mu_\Delta^\ominus$ is locally constant on $\{S_\infty^\ominus : S_\infty^\ominus \Delta \cap \Delta = 0\}$ (see §2.3) that the function $t \longmapsto \mu_\Delta^\ominus(S_\infty^\ominus(t))$ is constant, and hence

$$\mu_\Delta^\ominus(S_\infty^\ominus) = \mu_\Delta^\ominus(S_\infty^\ominus(0)) = \mu_\Delta^\ominus(S_\infty^\ominus(1)) = \mu_\Delta^\ominus(S_\infty'^\ominus)$$

which was to be proven. ∎

The following consequence of the result above shows that the indices ν and i_{CZ} coincide on their common domain of definition:

Corollary 4 *The restriction of the index ν to $\mathrm{Sp}^*(n)$ is the Conley–Zehnder index:*

$$\nu(S_\infty) = i_{CZ}(S_\infty) \quad if \quad \det(S - I) \neq 0.$$

Proof. The restriction of ν to $\mathrm{Sp}^*(n)$ satisfies the properties (CZ_1), (CZ_2), and (CZ_3) of the Conley–Zehnder index listed in §3.1; we showed that these properties uniquely characterize i_{CZ}. ∎

Let us prove a formula for the index of the product of two paths:

Proposition 5 *If S_∞, S'_∞, and $S_\infty S'_\infty$ are such that $\det(S - I) \neq 0$, $\det(S' - I) \neq 0$, and $\det(SS' - I) \neq 0$ then*

$$\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \tfrac{1}{2}\operatorname{sign}(M_S + M_{S'}) \qquad (30)$$

where M_S is the symplectic Cayley transform of S; in particular

$$\nu(S_\infty^r) = r\nu(S_\infty) + \tfrac{1}{2}(r-1)\operatorname{sign} M_S \qquad (31)$$

for every integer r.

Proof. In view of (27) and the product property (12) of the Maslov index we have

$$\nu(S_\infty S'_\infty) = \nu(S_\infty) + \nu(S'_\infty) + \tfrac{1}{2}\tau^\ominus(\Delta, S^\ominus\Delta, S^\ominus S'^\ominus\Delta)$$
$$= \nu(S_\infty) + \nu(S'_\infty) - \tfrac{1}{2}\tau^\ominus(S^\ominus S'^\ominus\Delta, S^\ominus\Delta, \Delta)$$

where $S^\ominus = I \oplus S$, $S'^\ominus = I \oplus S'$ and $\tau^\ominus$ is the signature on the symplectic space $(\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}, \sigma^\ominus)$. The condition $\det(SS' - I) \neq 0$ is equivalent to $S^\ominus S'^\ominus\Delta \cap \Delta = 0$ hence we can apply property *(i)* in Lemma 1 with $\ell = S^\ominus S'^\ominus\Delta$, $\ell' = S^\ominus\Delta$, and $\ell'' = \Delta$. The projection operator onto $S^\ominus S'^\ominus\Delta$ along Δ is easily seen to be

$$\operatorname{Pr}_{S^\ominus S'^\ominus\Delta,\Delta} = \begin{bmatrix} (I - SS')^{-1} & -(I - SS')^{-1} \\ SS'(I - SS')^{-1} & -SS'(I - SS')^{-1} \end{bmatrix}$$

hence $\tau^\ominus(S^\ominus S'^\ominus\Delta, S^\ominus\Delta, \Delta)$ is the signature of the quadratic form

$$Q(z) = \sigma^\ominus(\operatorname{Pr}_{S^\ominus S'^\ominus\Delta,\Delta}(z, Sz); (z, Sz))$$

that is, since $\sigma^\ominus = \sigma \ominus \sigma$:

$$Q(z) = \sigma((I - SS')^{-1}(I - S)z, z)) - \sigma(SS'(I - SS')^{-1}(I - S)z, Sz))$$
$$= \sigma((I - SS')^{-1}(I - S)z, z)) - \sigma(S'(I - SS')^{-1}(I - S)z, z))$$
$$= \sigma((I - S')(I - SS')^{-1}(I - S)z, z)).$$

In view of formula (21) in Lemma 2 we have

$$(I - S')(SS' - I)^{-1}(I - S) = (M_S + M_{S'})^{-1}J$$

hence

$$Q(z) = -\left\langle (M_S + M_{S'})^{-1}Jz, Jz \right\rangle$$

and the signature of Q is thus the same as that of

$$Q'(z) = -\left\langle (M_S + M_{S'})^{-1}z, z \right\rangle$$

that is $-\operatorname{sign}(M_S + M_{S'})$. This proves formula (30). Formula (31) follows from (30) by induction on r. $\blacksquare$

It is often deplored in the literature on Gutzwiller's formula (1) that it is not always obvious that the index μ_γ of the periodic orbit γ is independent on the choice of the origin of the orbit. Let us prove that this property always holds:

Proposition 6 *Let (f_t) be the flow determined by a (time-independent) Hamiltonian function on $\mathbb{R}^{2n}$ and $z \neq 0$ such that $f_T(z) = z$ for some $T > 0$. Let $z' = f_{t'}(z)$ for some t' and denote by $S_T(z) = Df_T(z)$ and $S_T(z') = Df_T(z')$ the corresponding monodromy matrices. Let $S_T(z)_\infty$ and $S_T(z')_\infty$ be the homotopy classes of the paths $t \longmapsto S_t(z) = Df_t(z)$ and $t \longmapsto S_t(z') = Df_t(z')$, $0 \leq t \leq T$. We have $\nu(S_T(z)_\infty) = \nu(S_T(z')_\infty)$.*

Proof. The monodromy matrices $S_T(z)$ and $S_T(z')$ are conjugate of each other; in fact (proof of Theorem 6 in [10]):

$$S_T(z') = S_{t'}(z')S_T(z)S_{t'}(z')^{-1};$$

since we will need to let t' vary we write $S_T(z') = S_T(z', t')$ so that

$$S_T(z', t') = S_{t'}(z')S_T(z)S_{t'}(z')^{-1}.$$

The paths $t \longmapsto S_t(z')$ and $t \longmapsto S_{t'}(z')S_t(z)S_{t'}(z')^{-1}$ being homotopic with fixed endpoints $S_T(z', t')_\infty$ is also the homotopy class of the path $t \longmapsto S_{t'}(z')S_t(z)S_{t'}(z')^{-1}$. We thus have, by definition (26) of ν,

$$\nu(S_T(z', t')_\infty) = \frac{1}{2}\mu_{\Delta_{t'}}^\ominus(S_T^\ominus(z)_\infty)$$

where we have set

$$\Delta_{t'} = (I \oplus S_{t'}(z')^{-1})\Delta \quad \text{and} \quad S_T^\ominus(z)_\infty = I_\infty \oplus S_T(z)_\infty.$$

Consider now the mapping $t' \longmapsto \mu_{\Delta_{t'}}^\ominus(S_T^\ominus(z)_\infty)$; we have

$$S_T^\ominus(z)\Delta_{t'} \cap \Delta_{t'} = \{z : Sz = z\}$$

hence the dimension of the intersection $S_T^\ominus(z)\Delta_{t'} \cap \Delta_{t'}$ remains constant as t' varies; in view of the topological property of the relative Maslov index the mapping $t' \longmapsto \mu_{\Delta_{t'}}^\ominus(S_T^\ominus(z)_\infty)$ is thus constant and hence

$$\nu(S_T(z', t')_\infty) = \nu(S_T(z', 0)_\infty) = \nu(S_T(z)_\infty)$$

which concludes the proof. ∎

17

3.4 Relation between ν and μ_{ℓ_P}

The index ν can be expressed in simple – and useful – way in terms of the Maslov index μ_{ℓ_P} on $\mathrm{Sp}_\infty(n)$. The following technical result will be helpful in establishing this relation. Recall that $S \in \mathrm{Sp}(n)$ is said to be "free" if we have $S\ell_P \cap \ell_P = 0$; this condition is equivalent to $\det B \neq 0$ when S is identified with the matrix

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{32}$$

in the canonical basis. The set of all free symplectic matrices is dense in $\mathrm{Sp}(n)$. The quadratic form W on $\mathbb{R}_x^n \times \mathbb{R}_x^n$ defined by

$$W(x, x') = \frac{1}{2} \langle Px, x \rangle - \langle Lx, x' \rangle + \frac{1}{2} \langle Qx', x' \rangle$$

where

$$P = DB^{-1}, \, L = B^{-1}, \, Q = B^{-1}A \tag{33}$$

then generates S in the sense that

$$(x, p) = S(x', p') \iff p = \partial_x W(x, x') \, , \, p' = -\partial_{x'} W(x, x').$$

We have:

Lemma 7 *Let $S = S_W \in \mathrm{Sp}(n)$ be given by (32). We have*

$$\det(S_W - I) = (-1)^n \det B \det(B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}) \tag{34}$$

that is:

$$\det(S_W - I) = (-1)^n \det(L^{-1}) \det(P + Q - L - L^T).$$

In particular the symmetric matrix

$$P + Q - L - L^T = DB^{-1} + B^{-1}A - B^{-1} - (B^T)^{-1}$$

is invertible.

Proof. Since B is invertible we can factorize $S - I$ as

$$\begin{bmatrix} A - I & B \\ C & D - I \end{bmatrix} = \begin{bmatrix} 0 & B \\ I & D - I \end{bmatrix} \begin{bmatrix} C - (D - I)B^{-1}(A - I) & 0 \\ B^{-1}(A - I) & I \end{bmatrix}$$

and hence

$$\det(S_W - I) = \det(-B) \det(C - (D - I)B^{-1}(A - I))$$
$$= (-1)^n \det B \det(C - (D - I)B^{-1}(A - I)).$$

Since S is symplectic we have $C - DB^{-1}A = -(B^T)^{-1}$ and hence

$$C - (D - I)B^{-1}(A - I)) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1};$$

the Lemma follows. ∎

Let us now introduce the notion of index of concavity of a Hamiltonian periodic orbit γ, defined for $0 \le t \le T$, with $\gamma(0) = \gamma(T) = z_0$. As t goes from 0 to T the linearized part $D\gamma(t) = S_t(z_0)$ goes from the identity to $S_T(z_0)$ (the monodromy matrix) in $\mathrm{Sp}(n)$. We assume that $S_T(z_0)$ is free and that $\det(S_T(z_0) - I) \ne 0$. Writing

$$S_t(z_0) = \begin{bmatrix} A(t) & B(t) \\ C(t) & D(t) \end{bmatrix}$$

we thus have $\det B(t) \ne 0$ in a neighborhood $[T - \varepsilon, T + \varepsilon]$ of the time T. The generating function

$$W(x, x', t) = \frac{1}{2} \langle P(t)x, x \rangle - \langle L(t)x, x' \rangle + \frac{1}{2} \langle Q(t)x', x' \rangle$$

(with $P(t)$, $Q(t)$, $L(t)$ defined by (33) thus exists for $T - \varepsilon \le t \le T + \varepsilon$. By definition Morse's index of concavity [17] (also see [18, 19]) of the periodic orbit γ is the index of inertia

$$\mathrm{Inert}\, W_{xx}'' = \mathrm{Inert}(P + Q - L - L^T)$$

of W_{xx}'', the matrix of second derivatives of the function $x \longmapsto W(x, x; T)$ (we have set $P = P(T)$, $Q = Q(T)$, $L = L(T)$).

Let us now prove the following essential result; recall that m_ℓ denotes the reduced Maslov index (15) associated to μ_ℓ:

Proposition 8 *Let $t \longmapsto S_t$ be a symplectic path, $0 \le t \le 1$. Let $S_\infty \in \mathrm{Sp}_\infty(n)$ be the homotopy class of that path and set $S = S_1$. If $\det(S - I) \ne 0$ and $S\ell_P \cap \ell_P = 0$ then*

$$\nu(S_\infty) = \frac{1}{2}(\mu_{\ell_P}(S_\infty) + \mathrm{sign}\, W_{xx}'') = m_{\ell_P}(S_\infty) - \mathrm{Inert}\, W_{xx}'' \qquad (35)$$

where $\mathrm{Inert}\, W_{xx}''$ is the index of concavity corresponding to the endpoint S of the path $t \longmapsto S_t$.

Proof. We will divide the proof in three steps. *Step 1.* Let $L \in \mathrm{Lag}^{\ominus}(4n)$. Using successively formulae (27) and (16) we have

$$\nu(S_\infty) = \frac{1}{2}(\mu_L^{\ominus}(S_\infty^{\ominus}) + \tau^{\ominus}(S^{\ominus}\Delta, \Delta, L) - \tau^{\ominus}(S^{\ominus}\Delta, S^{\ominus}L, L)). \qquad (36)$$

Choosing in particular $L = L_0 = \ell_P \oplus \ell_P$ we get

$$\begin{aligned}
\mu_{L_0}^{\ominus}(S_\infty^{\ominus}) &= \mu^{\ominus}((I \oplus S)_\infty(\ell_P \oplus \ell_P), (\ell_P \oplus \ell_P)) \\
&= \mu(\ell_{P,\infty}, \ell_{P,\infty}) - \mu(\ell_{P,\infty}, S_\infty \ell_{P,\infty}) \\
&= -\mu(\ell_{P,\infty}, S_\infty \ell_{P,\infty}) \\
&= \mu_{\ell_P}(S_\infty)
\end{aligned}$$

so that there remains to prove that

$$\tau^{\ominus}(S^{\ominus}\Delta, \Delta, L_0) - \tau^{\ominus}(S^{\ominus}\Delta, S^{\ominus}L_0, L_0) = -2\,\mathrm{sign}\,W_{xx}''.$$

Step 2. We are going to show that

$$\tau^{\ominus}(S^{\ominus}\Delta, S^{\ominus}L_0, L_0) = 0;$$

in view of the symplectic invariance and the antisymmetry of $\tau^{\ominus}$ this is equivalent to

$$\tau^{\ominus}(L_0, \Delta, L_0, (S^{\ominus})^{-1}L_0) = 0. \qquad (37)$$

We have

$$\Delta \cap L_0 = \{(0, p; 0, p) : p \in \mathbb{R}^n\}$$

and $(S^{\ominus})^{-1}L_0 \cap L_0$ consists of all $(0, p', S^{-1}(0, p''))$ with $S^{-1}(0, p'') = (0, p')$; since S (and hence S^{-1}) is free we must have $p' = p'' = 0$ so that

$$(S^{\ominus})^{-1}L_0 \cap L_0 = \{(0, p; 0, 0) : p \in \mathbb{R}^n\}.$$

It follows that we have

$$L_0 = \Delta \cap L_0 + (S^{\ominus})^{-1}L_0 \cap L_0$$

hence (37) in view of property *(ii)* in Lemma 1. *Step 3.* Let us finally prove that.

$$\tau^{\ominus}(S^{\ominus}\Delta, \Delta, L_0) = -2\,\mathrm{sign}\,W_{xx}'';$$

this will complete the proof of the proposition. The condition $\det(S - I) \neq 0$ is equivalent to $S^{\ominus}\Delta \cap \Delta = 0$ hence, using property *(i)* in Lemma 1:

$$\tau^{\ominus}(S^{\ominus}\Delta, \Delta, L_0) = -\tau^{\ominus}(S^{\ominus}\Delta, L_0, \Delta)$$

is the signature of the quadratic form Q on L_0 defined by

$$Q(0, p, 0, p') = -\sigma^\ominus(\mathrm{Pr}_{S^\ominus\Delta,\Delta}(0, p, 0, p'); 0, p, 0, p')$$

where

$$\mathrm{Pr}_{S^\ominus\Delta,\Delta} = \begin{bmatrix} (S-I)^{-1} & -(S-I)^{-1} \\ S(S-I)^{-1} & -S(S-I)^{-1} \end{bmatrix}$$

is the projection on $S^\ominus\Delta$ along Δ in $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$. It follows that the quadratic form Q is given by

$$Q(0, p, 0, p') = -\sigma^\ominus((I-S)^{-1}(0, p''), S(I-S)^{-1}(0, p''); 0, p, 0, p')$$

where we have set $p'' = p - p'$; by definition of $\sigma^\ominus$ this is

$$Q(0, p, 0, p') = -\sigma((I-S)^{-1}(0, p''), (0, p)) + \sigma(S(I-S)^{-1}(0, p''), (0, p')).$$

Let now M_S be the symplectic Cayley transform (19) of S; we have

$$(I-S)^{-1} = JM_S + \tfrac{1}{2}I \quad , \quad S(I-S)^{-1} = JM_S - \tfrac{1}{2}I$$

and hence

$$\begin{aligned}
Q(0, p, 0, p') &= -\sigma((JM_S + \tfrac{1}{2}I)(0, p''), (0, p)) + \sigma((JM_S - \tfrac{1}{2}I)(0, p''), (0, p')) \\
&= -\sigma(JM_S(0, p''), (0, p)) + \sigma(JM_S(0, p''), (0, p')) \\
&= \sigma(JM_S(0, p''), (0, p'')) \\
&= -\langle M_S(0, p''), (0, p'') \rangle.
\end{aligned}$$

Let us calculate explicitly M_S. Writing S in usual block-form we have

$$S - I = \begin{bmatrix} 0 & B \\ I & D-I \end{bmatrix} \begin{bmatrix} C - (D-I)B^{-1}(A-I) & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

that is

$$S - I = \begin{bmatrix} 0 & B \\ I & D-I \end{bmatrix} \begin{bmatrix} W''_{xx} & 0 \\ B^{-1}(A-I) & I \end{bmatrix}$$

where we have used the identity

$$C - (D-I)B^{-1}(A-I)) = B^{-1}A + DB^{-1} - B^{-1} - (B^T)^{-1}$$

which follows from the relation $C - DB^{-1}A = -(B^T)^{-1}$ (the latter is a rephrasing of the equalities $D^T A - B^T C = I$ and $D^T B = B^T D$, which follow from the fact that $S^T JS = S^T JS$ since $S \in \mathrm{Sp}(n)$). It follows that

$$
(S - I)^{-1} = \begin{bmatrix} (W''_{xx})^{-1} & 0 \\ B^{-1}(I - A)(W''_{xx})^{-1} & I \end{bmatrix} \begin{bmatrix} (I - D)B^{-1} & I \\ B^{-1} & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} (W''_{xx})^{-1}(I - D)B^{-1} & (W''_{xx})^{-1} \\ B^{-1}(I - A)(W''_{xx})^{-1}(I - D)B^{-1} + B^{-1} & B^{-1}(I - A)(W''_{xx})^{-1} \end{bmatrix}
$$

and hence

$$
M_S = \begin{bmatrix} B^{-1}(I - A)(W''_{xx})^{-1}(I - D)B^{-1} + B^{-1} & \frac{1}{2}I + B^{-1}(I - A)(W''_{xx})^{-1} \\ -\frac{1}{2}I - (W''_{xx})^{-1}(I - D)B^{-1} & -(W''_{xx})^{-1} \end{bmatrix}
$$

from which follows that

$$
Q(0, p, 0, p') = \langle (W''_{xx})^{-1} p'', p'' \rangle
$$
$$
= \langle (W''_{xx})^{-1}(p - p'), (p - p') \rangle .
$$

The matrix of the quadratic form Q is thus

$$
2 \begin{bmatrix} (W''_{xx})^{-1} & -(W''_{xx})^{-1} \\ -(W''_{xx})^{-1} & (W''_{xx})^{-1} \end{bmatrix}
$$

and this matrix has signature $2\,\mathrm{sign}(W''_{xx})^{-1} = 2\,\mathrm{sign}\,W''_{xx}$, proving the first equality (35); the second equality follows because $\mu_{\ell_P}(S_\infty) = 2m_{\ell_P}(S_\infty) - n$ since $S\ell_P \cap \ell_P = 0$ and the fact that W''_{xx} has rank n in view of Lemma 7. ∎

Remark 9 *Lemma 7 above shows that if S is free then we have*

$$
\frac{1}{\pi} \arg \det(S - I) \equiv n + \arg \det B + \arg \det W''_{xx} \pmod 2
$$
$$
\equiv n - \arg \det B + \arg \det W''_{xx} \pmod 2
$$

In [5, 6] we have shown that the reduced Maslov index $m_{\ell_P}(S_\infty)$ corresponds to a choice of $\arg \det B$ modulo 4; Proposition 8 thus justifies the following definition of the argument of $\det(S - I)$:

$$
\frac{1}{\pi} \arg \det(S - I) \equiv n - \nu(S_\infty) \pmod 4.
$$

That this is indeed the correct choice modulo 4 has been proven by other means (the Weyl theory of the metaplectic group) by one of us in a recent publication [11].

4   An Example

Let us begin with a very simple situation. Consider the one-dimensional harmonic oscillator with Hamiltonian function

$$H = \frac{\omega}{2}(p^2 + x^2);$$

all the orbits are periodic with period $2\pi/\omega$. The monodromy matrix is simply the identity: $\Sigma_T = I$ where

$$\Sigma_t = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}.$$

Let us calculate the corresponding index $\nu(\Sigma_\infty)$. The homotopy class of path $t \longmapsto \Sigma_t$ as t goes from 0 to $T = 2\pi/\omega$ is just the inverse of α, the generator of $\pi_1[\mathrm{Sp}(1)]$ hence $\nu(\Sigma_\infty) = -2$ in view of (29). If we had considered r repetitions of the orbit we would likewise have obtained $\nu(\Sigma_\infty) = -2r$.

Consider next a two-dimensional harmonic oscillator with Hamiltonian function

$$H = \frac{\omega_x}{2}(p_x^2 + x^2) + \frac{\omega_y}{2}(p_y^2 + y^2);$$

we assume that the frequencies ω_y, ω_x are incommensurate, so that the only periodic orbits are librations along the x and y axes. Let us focus on the orbit γ_x along the x axis; its prime period is $T = 2\pi/\omega_x$ and the corresponding monodromy matrix is

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\chi & 0 & \sin\chi \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\chi & 0 & \cos\chi \end{bmatrix} \quad , \quad \chi = 2\pi\frac{\omega_y}{\omega_x};$$

it is the endpoint of the symplectic path $t \longmapsto S_t$, $0 \le t \le 1$, consisting of the matrices

$$S_t = \begin{bmatrix} \cos 2\pi t & 0 & \sin 2\pi t & 0 \\ 0 & \cos\chi t & 0 & \sin\chi t \\ -\sin 2\pi t & 0 & \cos 2\pi t & 0 \\ 0 & -\sin\chi t & 0 & \cos\chi t \end{bmatrix}.$$

In Gutzwiller's formula (1) the sum is taken over periodic orbits, including their repetitions; we are thus led to calculate the Conley–Zehnder index of the path $t \longmapsto S_t$ with $0 \le t \le r$ where the integer r indicates the number of

repetitions of the orbit. Let us calculate the Conley–Zehnder index $\nu(\tilde{S}_{r,\infty})$ of this path. We have $S_t = \Sigma_t \oplus \tilde{S}_t$ where

$$\Sigma_t = \begin{bmatrix} \cos 2\pi t & \sin 2\pi t \\ -\sin 2\pi t & \cos 2\pi t \end{bmatrix} \quad , \quad \tilde{S}_t = \begin{bmatrix} \cos \chi t & \sin \chi t \\ -\sin \chi t & \cos \chi t \end{bmatrix};$$

in view of the additivity property of the relative Maslov index we thus have

$$\nu(S_{r,\infty}) = \nu(\Sigma_{r,\infty}) + \nu(\tilde{S}_{r,\infty})$$

where the first term is just

$$\nu(\Sigma_{r,\infty}) = -2r$$

in view of the calculation we made in the one-dimensional case with a different parametrization. Let us next calculate $\nu(\tilde{S}_{r,\infty})$. We will use formula (35) relating the index ν to the Maslov index via the index of concavity, so we begin by calculating the relative Maslov index

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = m(\tilde{S}_{r,\infty}\ell_{P,\infty}, \ell_{P,\infty}).$$

Here is a direct argument; in more complicated cases the formulas we proved in [10] are useful. When t goes from 0 to r the line $\tilde{S}_t \ell_P$ describes a loop in Lag(1) going from ℓ_P to $\tilde{S}_r \ell_P$. We have $\tilde{S}_t \in U(1)$; its image in $U(1, \mathbb{C})$ is $e^{-i\chi t}$ hence the Souriau mapping identifies $\tilde{S}_t \ell_P$ with $e^{-2i\chi t}$. It follows, using formula (9), that

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = \frac{1}{2\pi} \left(-2r\chi + i\,\mathrm{Log}(-e^{-2ir\chi}) \right) + \frac{1}{2}$$
$$= \frac{1}{2\pi} \left(-2r\chi + i\,\mathrm{Log}(e^{i(-2r\chi+\pi)}) \right) + \frac{1}{2}$$

The logarithm is calculated as follows: for $\theta \neq (2k+1)\pi \ (k \in \mathbb{Z})$

$$\mathrm{Log}\, e^{i\theta} = i\theta - 2\pi i \left[\frac{\theta + \pi}{2\pi} \right]$$

and hence
$$\mathrm{Log}(e^{i(-2r\chi+\pi)}) = -i(2r\chi + \pi + 2\pi \left[\frac{r\chi}{\pi} \right]);$$

it follows that the Maslov index is

$$m_{\ell_P}(\tilde{S}_{r,\infty}) = - \left[\frac{r\chi}{\pi} \right]. \tag{38}$$

To obtain $\nu(\tilde{S}_{r,\infty})$ we note that by (35)

$$\nu(\tilde{S}_{r,\infty}) = m_{\ell_P}(\tilde{S}_{1,\infty}) - \text{Inert } W''_{xx}$$

where Inert W''_{xx} is the concavity index corresponding to the generating function of $\tilde{S}_t$; the latter is

$$W(x, x', t) = \frac{1}{2\sin\chi t}((x^2 + x'^2)\cos\chi t - 2xx')$$

hence $W''_{xx} = -\tan(\chi t/2)$. We thus have, taking (38) into account,

$$\nu(\tilde{S}_{r,\infty}) = -\left[\frac{r\chi}{\pi}\right] - \text{Inert}\left(-\tan\frac{r\chi}{2}\right);$$

a straightforward induction on r shows that this can be rewritten more conveniently as

$$\nu(\tilde{S}_{r,\infty}) = -1 - 2\left[\frac{r\chi}{2\pi}\right].$$

Summarizing, we have

$$\nu(S_{r,\infty}) = \nu(\Sigma_{r,\infty}) + \nu(\tilde{S}_{r,\infty})$$
$$= -2r - 1 - 2\left[\frac{r\chi}{2\pi}\right])$$

hence the index in Gutzwiller's formula corresponding to the r-th repetition is

$$\mu_{x,r} = -\nu(S_{r,\infty}) = 1 + 2r + 2\left[\frac{r\chi}{2\pi}\right]$$

that is, by definition of χ,

$$\mu_{x,r} = 1 + 2r + 2\left[r\frac{\omega_y}{\omega_x}\right]$$

confirming the calculations in [1, 4, 20, 23].

Remark 10 *The calculations above are valid when the frequencies are incommensurate. If, say, $\omega_x = \omega_y$, the calculations are much simpler: in this case the homotopy class of the loop $t \longmapsto S_t$, $0 \le t \le 1$, is $\alpha^{-1} \oplus \alpha^{-1}$ and by the second formula (29),*

$$\mu_{x,r} = -\nu(S_{r,\infty}) = 4r$$

which is zero modulo 4 (cf. [20]).

References

[1] M Brack and R K Bhaduri 1997 *Semiclassical Physics* (Addison–Wesley)

[2] S E Cappell, R Lee and E Y Miller 1994 *Comm. Pure and Appl. Math.* **17** 121

[3] C Conley and E Zehnder 1984 *Comm. Pure and Appl. Math.* **37** 207

[4] S C Creagh, J M Robbins, and R G Littlejohn 1990 *Phys. Rev. A*, **42**(4) 1907

[5] M de Gosson 1990 *Ann. Inst. Fourier* **40**(3) 537

[6] M de Gosson 1992 *J. Geom. Phys.* **9** 255

[7] M de Gosson 1992 *J. Math. Pures et Appl.* **71** 429

[8] M de Gosson 1997 *Maslov Classes, Metaplectic Representation and Lagrangian Quantization* (*Research Notes in Mathematics 95*) (Berlin: Wiley–VCH)

[9] M de Gosson 2001 *The Principles of Newtonian and Quantum Mechanics: the Need for Planck's Constant h; with a foreword by B. Hiley* (London: Imperial College Press)

[10] M de Gosson and S de Gosson 2003 *J. Phys A: Math. and Gen.* **36** 615

[11] M de Gosson 2005 *Letters in Mathematical Physics* **72** 129

[12] M C Gutzwiller 1990 *Chaos in Classical and Quantum Mechanics* (Interdisciplinary Applied Mathematics) (Berlin: Springer)

[13] H. Hofer, K Wysocki, and E Zehnder 1995 *Geometric and Functional Analysis* **2**(5) 270

[14] J Leray 1981 *Lagrangian Analysis and Quantum Mechanics, a mathematical structure related to asymptotic expansions and the Maslov index* (Cambridge, MA: MIT Press,)

[15] G Lion, and M Vergne 1980 *The Weil representation, Maslov index and Theta series* (*Progress in mathematics* **6**) (Boston: Birkhäuser)

[16] B Mehlig and M Wilkinson 2001 *Ann. Phys.* **18**(10), 541

[17] M Morse 1935 *The Calculus of Variations in the Large* (Providence, R I: American Mathemtical Society)

[18] P Muratore-Ginnaneschi 2003 *Phys. Rep.* **383** 299

[19] R C Nostre Marques, P Piccione, and D Tausk 2001 in *Differential Geometry and its Applications* (Opava: Conference Proceedings)

[20] M Pletyukhov and M Brack 2003 *J. Phys A: Math. and Gen.* **36** 9449

[21] J M Robbin 1991 *Nonlinearity* **4** 343

[22] J M Souriau 1975 In: Group Theoretical Methods in Physics, Lecture Notes in Physics, Springer-Verlag, **50** 17

[23] A Sugita 2001 *Ann. Phys.* **288** 277

[24] C T C Wall 1969 *Invent. Math.* **7** 269.